

Gluing Busemann spaces

Bożena PIĄTEK and Alicja SAMULEWICZ

Abstract. We prove that the gluing of an arbitrary family of Busemann spaces along a singleton and the gluing of a metric tree and a Busemann space along a geodesic arc are Busemann spaces. We also demonstrate that the gluing of two Busemann spaces along a convex (compact) subspace is not necessarily a Busemann space.

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1. Introduction

One of the most successful methods leading to new examples of spaces are the gluing techniques. Namely, let us assume that $\{X_i \mid i \in \Gamma\}$ is a set of (metric) spaces and A is a set such that each X_i has a subset A_i which is isomorphic to A . Then there is a natural way to introduce the topological structure on the gluing of X_i along the set A , i.e., the amalgamation $\coprod_A X_i$ (the precise definitions and most important properties the reader may find in Section 2). Clearly, if all X_i are metric spaces and the isomorphisms between A and A_i are isometries, the amalgamation $\coprod_A X_i$ is also a metric space. It seems to be much more interesting to raise the related question which properties of $\coprod_A X_i$ can be deduced from the facts about X_i .

It is known that the gluing of geodesic spaces does not have to lead to the same type of space until the set A is not proper (see [1]). However, some additional assumptions imposed on X_i may do not guarantee that the amalgamation satisfies the same property even if A is compact and convex set. For instance in [9] the first author gave the example showing that gluing of two hyperconvex spaces along the metric segment $[a, b]$ is not necessary a hyperconvex space. At the same time in [5] we have shown that the gluing along a point of two complete metric trees – spaces belonging

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to a subclass of hyperconvex ones is also a hyperconvex space (more on hyperconvex spaces and their amalgamation the reader may find in further generalization of this fact to the case of hyperconvex spaces the reader may [4,6,7,9]). The main goal of this paper is to generalize the result about gluing of metric trees. Moreover, we will show how some special kind of convexity of the metric (so-called Busemann spaces) impact on the properties of amalgamation. In particular, the gluing of two Busemann spaces along a metric segment does not have to be a Busemann space (Section 5). This fact is really worth of emphasizing because the same gluing of elements of a wide subclass of Busemann spaces gives completely different result. Namely, it is known that the amalgamation of $CAT(\kappa)$ spaces along a convex complete subspace is a $CAT(\kappa)$ space (see [1]).

Not counting the introduction, our paper consists of four sections. In Section 2 we gathered the definitions and the most important facts concerning Busemann and $CAT(0)$ spaces. Our intention is to highlight the differences in methods and approach to both classes of spaces, so we decided to discuss in detail a number of classic results, paying special attention to the ones concerning gluing. Most of them are accompanied with their proofs provided for convenience of the reader.

The other sections, containing new results only, are devoted solely to the properties of gluing of Busemann spaces along some simple subspaces. In Section 3 we demonstrate that the gluing of an arbitrary family of Busemann spaces along a singleton is a Busemann space, while in Section 4 we deal with the gluing of a metric tree and a Busemann space along a geodesic arc. Section 5 contains examples showing that the gluing of two Busemann spaces along their convex subspaces is not always a Busemann space, even if the considered subspace is a metric segment. This is another evidence of the fact that the geometric structure of $CAT(0)$ spaces is extraordinary even in the class of Busemann spaces.

2. Preliminaries

Let (X, d) be a metric space. By a *path* in X joining two points $x, y \in X$ we mean a continuous map $\gamma : [a, b] \rightarrow X$ with $a < b$, $\gamma(a) = x$ and $\gamma(b) = y$. The *length* of the path γ is the value of the following formula:

$$L(\gamma) = \sup_{\sigma} \sum_{i=0}^{|\sigma|-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all the divisions $\sigma : a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$ and $|\sigma| = n$ denotes the number of subintervals determined by σ . The length of a path is invariant under the change of parameter, i.e. if $\phi : [c, d] \rightarrow [a, b]$ is (weakly) monotone surjective map then $L(\gamma) = L(\gamma \circ \phi)$.

A path $\gamma : [a, b] \rightarrow X$ is a *geodesic* (or a *geodesic path*) in X provided that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. If every two points of a metric space can be joined by a geodesic then the space is said to be a *geodesic space*. A *geodesic segment* is the image of a geodesic path; geodesic segments are subsets of a space while geodesic paths are maps and can be considered as parametrizations of geodesic segments. In

the following all the spaces under consideration will be assumed to be geodesic and $[x, y]$ will denote a geodesic segment with ends x and y , although in general it may be confusing as a geodesic segment is not necessarily uniquely determined by its ends. If for any pair of points there exists exactly one geodesic segment joining them then the space is called *uniquely geodesic*. In this case $(1 - t)x + ty$ will stand for the point in $[x, y]$ whose distance to x is $td(x, y)$ and the distance to y is $(1 - t)d(x, y)$.

2.1. Busemann spaces

Suppose that A and B are geodesic segments with ends x_0, x_1 and y_0, y_1 , respectively. Then for each $t \in [0, 1]$ one can find the points $x_t \in A$ and $y_t \in B$ such that $d(x_0, x_t) = td(x_0, x_1)$ and $d(y_0, y_t) = td(y_0, y_1)$. If a metric space (X, d) is geodesic and for any geodesic segments A, B as above and every $t \in [0, 1]$ we have

$$d(x_t, y_t) \leq (1 - t)d(x_0, y_0) + td(x_1, y_1) \quad (1)$$

then (X, d) is called a *Busemann space*. Clearly Busemann spaces are uniquely geodesic. Indeed, (1) applied to $x_0 = y_0$ and $x_1 = y_1$ implies that $x_t = y_t$ for every $t \in [0, 1]$, so there is exactly one geodesics arc joining x_0 and x_1 .

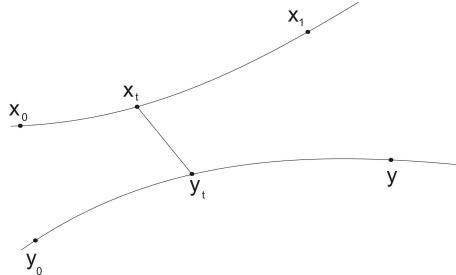


Fig. 1. Geodesic arcs $[x_0, x_1]$, $[y_0, y_1]$ and point x_t, y_t

The condition (1) in the definition of Busemann space can be substituted by the following one:

$$d(x_{1/2}, y_{1/2}) \leq \frac{1}{2} (d(x_0, y_0) + d(x_1, y_1)) \quad (2)$$

for arbitrary two geodesic segments with ends x_0, x_1 and y_0, y_1 .

Even the latter condition can be weakened: it is enough to verify (1) or (2) for $x_0 = y_0$ only. It turns out that X is a Busemann space if and only if

$$d(y_{1/2}, z_{1/2}) \leq \frac{1}{2}d(y, z) \quad (3)$$

for any geodesic segments $[x, y]$, $[x, z]$ and their midpoints $y_{1/2}, z_{1/2}$.

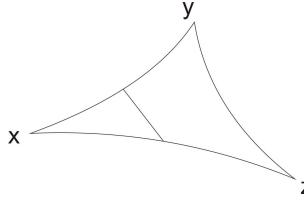


Fig. 2. Geodesic segments $[x, y]$, $[x, z]$ and their midpoints $y_{1/2}, z_{1/2}$

The proofs of the above statements and more conditions equivalent to the definition of Busemann space can be found in [8], Proposition 8.1.2.

Example 2.1. The Euclidean space \mathbb{R}^n is a Busemann space. Let B' and C' be the midpoints of segments \overline{AB} and \overline{AC} , respectively. By the theorem of Thales, $\|\overline{B'C'}\| = \frac{1}{2}\|\overline{BC}\|$, in particular (3) is satisfied.

Recognizing Busemann spaces among normed linear spaces is easier than in general case. The proposition below provides simple criteria which can be considered instead of verifying (1). Recall that a normed linear space $(V, \|\cdot\|)$ is *strictly convex* provided that for all $x, y \in V$, $x \neq y$, satisfying $\|x\| = \|y\| = 1$ and for all $t \in (0, 1)$ we have $\|(1-t)x + ty\| < 1$.

Proposition 2.2. ([8], Proposition 8.1.6) *Let $(V, \|\cdot\|)$ be a normed linear space, equipped with the metric determined by its norm. Then the following conditions are equivalent:*

- (1) V is a Busemann space;
- (2) V is uniquely geodesic;
- (3) V is strictly convex.

Example 2.3. Consider $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|x\|_1 = \sum_{i=1}^n |x_i|$, and $\|x\|_\infty = \sup\{|x_i| : i = 1, \dots, n\}$. None of these spaces is strictly convex, so they are not Busemann spaces either.

Let e_1, \dots, e_n denote the vectors of the standard basis in \mathbb{R}^n . Then $\|e_1\|_1 = \|e_2\|_1 = 1$ and $\|te_1 + (1-t)e_2\|_1 = 1$ for all $t \in [0, 1]$. Putting $x = e_1$ and $y = e_1 + e_2$ we have $\|x\|_\infty = \|y\|_\infty = 1$ and $\|tx + (1-t)y\|_\infty = 1$ for each $t \in [0, 1]$. According to Proposition 2.2 neither $(\mathbb{R}^n, \|\cdot\|_1)$ nor $(\mathbb{R}^n, \|\cdot\|_\infty)$ is not uniquely geodesic.

Example 2.4. If $p \in (1, \infty)$ then $(\mathbb{R}^n, \|\cdot\|_p)$ is a Busemann space. Assume that x and y in \mathbb{R}^n are not collinear. By the Minkowski inequality, for each $t \in (0, 1)$ we have $\|(1-t)x + ty\|_p < (1-t)\|x\|_p + t\|y\|_p$, hence $(\mathbb{R}^n, \|\cdot\|_p)$ is a Busemann space.

Example 2.5. Let X be a unitary space, real or complex, equipped with the norm $\|\cdot\|$ determined by an inner product $\langle \cdot, \cdot \rangle$. Assume that $\|x\| = \|y\| = 1$ and $x \neq y$. Then for each $t \in (0, 1)$ we have

$$\begin{aligned} \|(1-t)x + ty\|^2 &= (1-t)^2\|x\|^2 + 2t(1-t)\operatorname{Re}\langle x, y \rangle + t^2\|y\|^2 \\ &\leq (1-t)^2\|x\|^2 + 2t(1-t)|\langle x, y \rangle| + t^2\|y\|^2. \end{aligned} \quad (4)$$

By the Cauchy-Schwarz inequality, if x and y are not collinear then $|\langle x, y \rangle| < \|x\| \cdot \|y\|$ and, consequently, $\|(1-t)x + ty\|^2 < ((1-t)\|x\| + t\|y\|)^2 = 1$. If x and y are

linearly dependent, seeing that $\|x\| = \|y\| = 1$ and $x \neq y$, we have $y = -x$. Thus $\|(1-t)x + ty\| = |1-2t| < 1$ for every $t \in (0, 1)$. Therefore every unitary space is strictly convex and, according to Proposition 2.2, a Busemann space.

2.2. CAT(0) spaces

A special class of Busemann spaces constitute CAT(0) spaces, named by M. Gromov in honour of E. Cartan, A. D. Alexandrov and V. A. Toponogov. Actually, one can define CAT(κ) spaces for any $\kappa \in \mathbb{R}$, where the parameter κ stands for the curvature of so called comparison space, but we discuss here the case of $\kappa = 0$ only. It is worth mentioning that Alexandrov's Lemma 2.20, Theorems 1 and 2 below are particular versions of more general theorems which are valid for all $\kappa \in \mathbb{R}$ and can be found e.g. in [1]. Since the proofs for $\kappa = 0$ are significantly simpler than in general case, we provide them for the reader's convenience.

Let p, q, r be three distinct points in a geodesic space (X, d) . Then there exist points $\bar{p}, \bar{q}, \bar{r}$ in the Euclidean plane \mathbb{R}^2 such that $d(p, q) = \|\bar{p} - \bar{q}\|$, $d(p, r) = \|\bar{p} - \bar{r}\|$ and $d(q, r) = \|\bar{q} - \bar{r}\|$. Moreover, the triple $\bar{p}, \bar{q}, \bar{r}$ is unique up to an isometry of the plane.

The union of geodesic arcs $[p, q]$, $[p, r]$ and $[q, r]$ is called a *geodesic triangle* in X with vertices p, q, r and sides $[p, q]$, $[p, r]$ and $[q, r]$. The points $\bar{p}, \bar{q}, \bar{r}$ are the vertices of a triangle $\Delta(\bar{p}, \bar{q}, \bar{r}) \subset \mathbb{R}^2$, called a *comparison triangle* for the geodesic triangle $\Delta(p, q, r) \subset X$. Observe that $\Delta(\bar{p}, \bar{q}, \bar{r})$ reduces to a segment if and only if $\Delta(p, q, r)$ is a geodesic segment itself. A point $\bar{x} \in \Delta(\bar{p}, \bar{q}, \bar{r})$ is said to be a *comparison point* for $x \in [p, q]$ provided that $\|\bar{x} - \bar{p}\| = d(x, p)$ and $\|\bar{x} - \bar{q}\| = d(x, q)$.

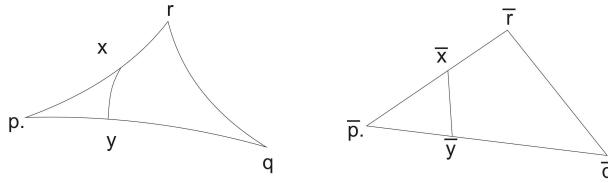


Fig. 3. Triangles $\Delta(p, q, r)$ and $\Delta(\bar{p}, \bar{q}, \bar{r})$

A geodesic triangle $\Delta(p, q, r)$ satisfies the *CAT(0) inequality* if for all $x, y \in \Delta(p, q, r)$ and all comparison points $\bar{x}, \bar{y} \in \Delta(\bar{p}, \bar{q}, \bar{r})$ we have

$$d(x, y) \leq \|\bar{x} - \bar{y}\|. \quad (5)$$

If all geodesic triangles in (X, d) satisfy the CAT(0) inequality then (X, d) is said to be a *CAT(0) space*. Obviously all CAT(0) spaces are Busemann spaces and thus uniquely geodesic spaces.

Let $\gamma : [0, a] \rightarrow X$ and $\eta : [0, b] \rightarrow X$ be two geodesic paths issuing from the same point $\gamma(0) = \eta(0)$. Given $t \in (0, a]$ and $s \in (0, b]$, we consider the comparison triangle $\bar{\Delta} = \Delta(\gamma(0), \gamma(t), \eta(s))$ and the comparison angle $\bar{\angle}_{\gamma(0)}(\gamma(t), \eta(s))$. By the *Alexandrov*

angle between the geodesic paths γ and η we mean the number $\angle(\gamma, \eta) \in [0, \pi]$ defined as follows:

$$\angle(\gamma, \eta) = \limsup_{s, t \rightarrow 0} \overline{\angle}_{\gamma(0)}(\gamma(t), \eta(s)) = \lim_{\varepsilon \rightarrow 0} \sup_{s, t \in (0, \varepsilon)} \overline{\angle}_{\gamma(0)}(\gamma(t), \eta(s)). \quad (6)$$

Applying the law of cosines one can express the Alexandrov angle $\angle(\gamma, \eta)$ in terms of distances in (X, d) :

$$\cos \angle(\gamma, \eta) = \liminf_{s, t \rightarrow 0} \frac{1}{2st} (s^2 + t^2 - d(\gamma(t), \eta(s))^2). \quad (7)$$

The statement below provides a useful characterization of CAT(0) spaces. (Compare [1], Chapter II, Proposition 1.7.)

Proposition 2.6. *A geodesic space X is a CAT(0) space if and only if for any geodesic triangle in X every Alexandrov angle between its sides is no greater than the angle between the corresponding sides of its comparison triangle in \mathbb{R}^2 .*

Example 2.7. A uniquely geodesic space T is an \mathbb{R} -tree provided that for each $x, y, z \in T$ the condition $[x, y] \cap [y, z] = \{y\}$ implies $[x, y] \cup [y, z] = [x, z]$. As the name suggests, trees, i.e. metric graphs without cycles, are in particular \mathbb{R} -trees. Let p, q, r be three distinct points in an \mathbb{R} -tree T . If one of them, e.g. q , lies on the geodesic segment joining the other two points then the geodesic triangle $\triangle(p, q, r)$ and its comparison triangle $\overline{\triangle} = \triangle(\overline{p}, \overline{q}, \overline{r})$ degenerate to the segments $[p, r]$ and $[\overline{p}, \overline{r}]$, respectively, and their corresponding angles are equal. Otherwise all the Alexandrov angles of the geodesic triangle $\triangle(p, q, r)$ equal 0 and they are smaller than the angles of the comparison triangle $\overline{\triangle}$ in \mathbb{R}^2 . Therefore all \mathbb{R} -trees are CAT(0) spaces and, in particular, Busemann spaces.

Example 2.8. Let X be a unitary space, real or complex. Computing the angles of a geodesic triangle $\triangle \subset X$ with the formula (7) we get the same results as for the corresponding angles of its comparison triangle $\overline{\triangle} \subset \mathbb{R}^2$. Therefore every unitary space is a CAT(0) space.

Proposition 2.9. ([1], Chapter II, 1.14) *A real normed linear space is a CAT(0) space if and only if it is unitary, i.e. its norm is determined by an inner product.*

Proof. Every unitary space is a CAT(0) space, so it remains to show the converse implication.

Assume that a real normed linear space X is a CAT(0) space. It suffices to show that X satisfies the parallelogram law.

Let v, w be two linearly independent vectors in X . First we show that for all $s, t \in (0, 1]$ the comparison angles $\overline{\angle}([\mathbf{0}, sv], [\mathbf{0}, tw])$ are equal to the Alexandrov angle $\alpha = \angle([\mathbf{0}, v], [\mathbf{0}, w])$. Let $M = \max\{\|v\|, \|w\|\}$, $u_1 = \frac{v}{\|v\|}$, $u_2 = \frac{w}{\|w\|}$, and let $\gamma_i : [0, M] \rightarrow X$, $i \in \{0, 1\}$, be the geodesic paths given by $\gamma_i(t) = t \cdot u_i$. Recall that

$$\alpha = \angle(\gamma_1, \gamma_2) = \angle([\mathbf{0}, su_1], [\mathbf{0}, tu_2]) \quad (8)$$

for all $s, t \in (0, M]$. Observe that if the limit in the definition (6) of the Alexandrov angle exists then for all $\varepsilon \in (0, M]$ we have

$$\sup_{s,t \in (0,\varepsilon)} \overline{\angle}_0(\gamma_1(s), \gamma_2(t)) \leq \sup_{s,t \in (0,M]} \overline{\angle}_0(\gamma_1(s), \gamma_2(t)). \quad (9)$$

On the other hand, by linearity, $\overline{\angle}_0(\gamma_1(\frac{\varepsilon}{2M} \cdot s), \gamma_2(\frac{\varepsilon}{2M} \cdot t)) = \overline{\angle}_0(\gamma_1(s), \gamma_2(t))$, so

$$\sup_{s,t \in (0,\varepsilon)} \overline{\angle}_0(\gamma_1(s), \gamma_2(t)) \geq \sup_{s,t \in (0,M]} \overline{\angle}_0(\gamma_1(s), \gamma_2(t)). \quad (10)$$

Therefore

$$\alpha = \angle(\gamma_1, \gamma_2) = \sup_{s,t \in (0,M]} \overline{\angle}_0(\gamma_1(s), \gamma_2(t)). \quad (11)$$

Clearly $\overline{\angle}_0(\gamma_1(s), \gamma_2(t)) \leq \alpha$ for any $s, t \in (0, M]$. Since X is a CAT(0) space and (8) holds, we have also $\angle_0(\gamma_1(s), \gamma_2(t)) \leq \overline{\angle}_0(\gamma_1(s), \gamma_2(t))$. Thus

$$\alpha = \overline{\angle}([\mathbf{0}, su_1], [\mathbf{0}, tu_2]) \quad (12)$$

for all $s, t \in (0, M]$.

Now consider the triangle $\triangle(\mathbf{0}, v, v+w)$ in X . According to (12), we have

$$\overline{\angle}([\mathbf{0}, v], [\mathbf{0}, v+w]) = \overline{\angle}([\mathbf{0}, v], [\mathbf{0}, \frac{1}{2}(v+w)]).$$

Applying the law of cosines to the comparison triangles $\overline{\triangle}(\mathbf{0}, v, v+w)$ and $\overline{\triangle}(\mathbf{0}, v, \frac{v+w}{2})$ we obtain

$$\|v+w\|^2 + \|v\|^2 - \|w\|^2 = \frac{1}{2}(\|v+w\|^2 + 4\|v\|^2 - \|v-w\|^2),$$

which is equivalent to

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

□

Corollary 2.10. *Let $p \in [1, \infty]$. The spaces ℓ_p and $(\mathbb{R}^n, \|\cdot\|_p)$ are CAT(0) spaces if and only if $p = 2$.*

2.3. Gluing of spaces

Let $(X_\lambda, d_\lambda)_{\lambda \in \Lambda}$ be a family of metric spaces with closed subspaces $A_\lambda \subset X_\lambda$. Assume that all the spaces A_λ are isometric to a metric space A and denote the appropriate isometries by $i_\lambda : A \rightarrow A_\lambda$. Let $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ be the disjoint union of the spaces X_λ , $\lambda \in \Lambda$. Define an equivalence relation \sim in $\bigsqcup_{\lambda \in \Lambda} X_\lambda$ as follows: $x \sim y$ if and only if $x = y$ or $x = i_\alpha(a)$ and $y = i_\beta(a)$ for some $a \in A$ and $\alpha, \beta \in \Lambda$. The quotient space $X = \coprod_A X_\lambda$ is called the *gluing* or *amalgamation* of the spaces X_λ along A . We identify each X_λ with its image in X . The metric d in X , given by the formula

$$d(x, y) = \begin{cases} d_\lambda(x, y) & \text{if } x, y \in X_\lambda \text{ for some } \lambda \in \Lambda \\ \inf_{a \in A} \{d_\alpha(x, i_\alpha(a)) + d_\beta(i_\beta(a), y)\} & \text{otherwise, for } x \in X_\alpha, y \in X_\beta, \end{cases} \quad (13)$$

coincides with d_λ on every subspace X_λ .

Remark 2.11. The amalgamation of two geodesic spaces is not always a geodesic space, even if both spaces are CAT(0).

Example 2.12. The set $Y = (0, 1] \times [-1, 1]$ equipped with the Euclidean metric is a CAT(0) space (in particular, it is uniquely geodesic) and $A = \{(x, y) \in Y : y = x\}$ is its closed subset. Let $X_1 = Y \times \{1\}$, $A_1 = A \times \{1\}$, $X_2 = Y \times \{2\}$ and $A_2 = A \times \{2\}$. The amalgamation $X = X_1 \amalg_A X_2$ is not a geodesic space because $d((1, -1, 1), (1, -1, 2)) = 2\sqrt{2}$ and every arc joining $(1, -1, 1)$ and $(1, -1, 2)$ in X has length greater than $2\sqrt{2}$.

Lemma 2.13. ([1], Chapter II.11.1) *Let X_1 and X_2 be CAT(0) spaces and let A_1, A_2 be convex subspaces of X_1, X_2 , respectively. If A_1 and A_2 are isometric to a complete metric space A then $X_1 \amalg_A X_2$ is a geodesic space.*

Moreover, if $x \in X_1 \setminus A_1$ and $y \in X_2 \setminus A_2$ then there exists a point $z \in A$ such that $[x, y] = [x, z] \cup [z, y]$ with $[x, z] \subset X_1$ and $[z, y] \subset X_2$.

Proof. It suffices to show that for all $x \in X_1 \setminus A_1$ and $y \in X_2 \setminus A_2$ there exists $z \in A_1$ such that $d(x, y) = d(x, z) + d(z, y)$. By the definition (13) of the metric d , for every $n \in \mathbb{N}$ there exists a point $z_n \in A$ such that $d(x, z_n) + d(z_n, y) \leq d(x, y) + \frac{1}{n}$. Note that the sequence of the numbers $d(x, z_n)$ is bounded so it contains a converging subsequence. Without loss of generality we can assume that $d(x, z_n) \rightarrow s$ as $n \rightarrow \infty$. Thus $d(z_n, y) \rightarrow u = d(x, y) - s$ as $n \rightarrow \infty$. Given $\epsilon > 0$, let N be such a number that $\max\{|d(x, z_n) - s|, |d(y, z_n) - u|\} < \epsilon$ for all $n > N$. Fix $n, m > N$, $n \neq m$, and denote by p the midpoint of the geodesic segment $[z_n, z_m]$. By convexity of A_2 we get $[z_n, z_m] \subset A_2$, and, in particular, $p \in X_2$. Recall that X_2 is a Busemann space, so (1) yields

$$d(y, p) \leq \max\{d(z_n, y), d(z_m, y)\} \leq u + \epsilon \quad (14)$$

and thereby

$$d(x, p) \geq s - \epsilon. \quad (15)$$

Let $\bar{x}, \bar{z}_n, \bar{z}_m$ be the vertices of a comparison triangle $\bar{\triangle}$. Then $\|\bar{x} - \bar{p}\|$ is the length of its median and

$$\|\bar{x} - \bar{p}\|^2 = \frac{1}{2}\|\bar{x} - \bar{z}_n\|^2 + \frac{1}{2}\|\bar{x} - \bar{z}_m\|^2 - \frac{1}{4}\|\bar{z}_n - \bar{z}_m\|^2. \quad (16)$$

(The above equality can be derived e.g. from the law of cosines.) By the CAT(0) inequality (5) in X_1 we have

$$d(x, p) \leq \|\bar{x} - \bar{p}\|. \quad (17)$$

By (16) and (15) we get

$$\begin{aligned} d(z_n, z_m)^2 &= \|\bar{z}_n - \bar{z}_m\|^2 = 2d(x, z_n)^2 + 2d(x, z_m)^2 - 4\|\bar{x} - \bar{p}\|^2 \\ &\leq 2d(x, z_n)^2 + 2d(x, z_m)^2 - 4d(x, p)^2 \leq 4(s + \epsilon)^2 - 4(s - \epsilon)^2 = 16s\epsilon. \end{aligned} \quad (18)$$

Therefore (z_n) is a Cauchy sequence in a complete space A . Its limit $z_0 \in A$ satisfies the condition $d(x, z_0) + d(z_0, y) = d(x, y)$ and $[x, z_0] \cup [z_0, y]$ is a geodesic arc in X with $[x, z_0] \subset X_1$, $[z_0, y] \subset X_2$. \square

Remark 2.14. Note that the assumptions of completeness and convexity of A are essential. The space A described in Example 2.13 is isometric to the closed and convex subsets A_1, A_2 of the CAT(0) spaces Y_1, Y_2 , respectively, but $X_1 \amalg_A X_2$ fail to be a geodesic space. On the other hand, consider a tree $T = \bigcup_{n \in \mathbb{N}} J_n$ with countably many edges J_n issuing from a common point p and pairwise disjoint apart of it. Let $J_n = [p, q_n]$, $n \in \mathbb{N}$, be a segment of length $1 + \frac{1}{n}$. Then T equipped with the arclength metric is a CAT(0) space and $A = \{q_n : n \in \mathbb{N}\}$ is its complete subspace. If $X_1 = X_2 = T$ then $X_1 \amalg_A X_2$ is not a geodesic space. Indeed, the distance between the copies $p_1 \in X_1$ and $p_2 \in X_2$ of the point p equals 2 while every arc joining them has length $2 + \frac{2}{n}$ for some $n \in \mathbb{N}$.

Remark 2.15. Note that the assumption that X_1 and X_2 are CAT(0) was used in the proof of Lemma 2.13 in order to show that (z_n) contains a Cauchy sequence. To achieve the same result one can assume that X_1, X_2 are geodesic spaces and A is compact or at least proper, i.e. every closed ball in A is compact.

Combining the latter with the fact that the natural embedding of $X_{\lambda_1} \amalg_A X_{\lambda_2}$ into $\coprod_A X_\lambda$ is an isometry we get the following statement concerning gluing of geodesic spaces.

Corollary 2.16. *Let $(X_\lambda, d_\lambda)_{\lambda \in \Lambda}$ be a family of geodesic spaces with closed subspaces $A_\lambda \subset X_\lambda$ isometric to a proper space A . Then the space $X = \coprod_A X_\lambda$ obtained by gluing X_λ along A is geodesic.*

Under additional assumptions we obtain stronger conclusions concerning the structure of $X_1 \amalg_A X_2$.

Observation 2.17. *Assume that X_1 and X_2 are uniquely geodesic spaces such that for all $x, y, z \in X_i$, $i \in \{1, 2\}$, and $t \in (0, 1)$ the condition $x \neq y, z \notin [x, y]$ implies*

$$d((1-t)x + ty, z) < (1-t)d(x, z) + td(y, z), \quad (19)$$

where $(1-t)x + ty$ denotes the point in $[x, y]$ with $d((1-t)x + ty, x) = td(x, y)$ and $d((1-t)x + ty, y) = (1-t)d(x, y)$. Let A_1, A_2 be closed convex subspaces of X_1, X_2 isometric to a space A . If the space $X_1 \amalg_A X_2$ is geodesic then it is uniquely geodesic.

Proof. Suppose that there are two geodesic arcs joining the same pair of points x, y in $X_1 \amalg_A X_2$. Clearly x, y cannot belong to the same space X_i , $i \in \{1, 2\}$. Assume that $x \in X_1, y \in X_2$ and $a, b \in A$ satisfy the following equality:

$$d(x, a) + d(a, y) = d(x, b) + d(b, y) = d(x, y).$$

If $a \neq b$ then $c = \frac{1}{2}a + \frac{1}{2}b \in A$ and, by (19), $d(x, c) + d(c, y) < \frac{1}{2}d(x, a) + \frac{1}{2}d(x, b) + \frac{1}{2}d(a, y) + \frac{1}{2}d(b, y) = d(x, y)$. \square

Remark 2.18. All strictly convex Banach spaces and uniformly convex metric spaces satisfy (19) (see [2, 3]).

Corollary 2.19. *Let X_1 and X_2 be normed linear Busemann spaces with closed convex subspaces A_1, A_2 isometric to a space A . If the space $X_1 \amalg_A X_2$ is geodesic then it is uniquely geodesic.*

Proof. The proof follows from Proposition 2.2. Observe that strict convexity of a normed linear space implies that $\|tu + (1-t)v\| < t\|u\| + (1-t)\|v\|$ for any non-zero elements u, v and each number $t \in (0, 1)$. \square

We make use of the latter considerations in Section 5, however now we return to $\text{CAT}(0)$ spaces. In the following we employ a weaker version of Alexandrov's lemma ([1], Chapter I.2., 2.16) concerning geodesic triangles in surfaces M_κ^2 with constant curvature $\kappa \in \mathbb{R}$. If $\kappa = 0$ then M_κ^2 is the Euclidean plane \mathbb{R}^2 .

Lemma 2.20 (Alexandrov's Lemma). *Consider triangles $\Delta_1 = \Delta(A, B_1, C)$ and $\Delta_2 = \Delta(A, B_2, C)$ in \mathbb{R}^2 with B_1 and B_2 lying on opposite sides of the line through A and C . Let $\alpha_i, \beta_i, \gamma_i$ be the angles of Δ_i at the vertices A, B_i, C , respectively, for $i \in \{1, 2\}$. Assume that*

$$\gamma_1 + \gamma_2 \geq \pi.$$

Let $\overline{\Delta} = \Delta(A, B_1, D)$ where $|AD| = |AB_2|$ and $|B_1D| = |B_1C| + |B_2C|$. Let α, β, γ be the angles of $\overline{\Delta}$ at the vertices A, B_1, D . Then

$$\alpha \geq \alpha_1 + \alpha_2, \quad \beta \geq \beta_1, \quad \gamma \geq \beta_2.$$

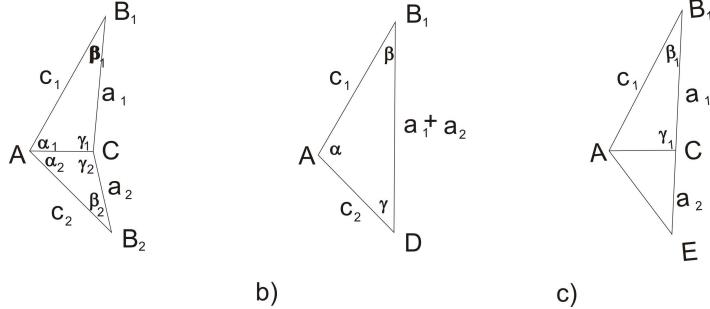


Fig. 4. a) Adjacent triangles Δ_1 and Δ_2 , b) triangle $\overline{\Delta}$, c) auxiliary triangle $\Delta(A, B_1, E)$

Proof. Let E be the only point such that $|CE| = |CB_2|$ and C lies on the segment B_1E in \mathbb{R}^2 . Since $\gamma_1 + \gamma_2 \geq \pi$, the angle at C between CA and CE is no greater than the angle between CA and CB_2 . Because $|CE| = |CB_2|$, the law of cosines applied to $\Delta(ACE)$ and $\Delta(ACB_2)$ yields $|AE| \leq |AB_2|$. Therefore $|B_1C| + |B_2C| = |B_1C| + |CE| = |B_1E| \leq |AB_1| + |AE| \leq |AB_1| + |AB_2|$ and a triangle $\Delta(A, B_1, D)$ with $|AD| = |AB_2|$ and $|B_1D| = |B_1C| + |B_2C|$ is constructible.

Clearly $|B_1B_2| \leq |BD|$, so $\alpha_1 + \alpha_2 \leq \alpha$. To show that $\beta_1 \leq \beta$ consider again the triangles $\Delta(A, B_1, E)$ and $\Delta(A, B_1, D)$. Recall that $|AE| \leq |AB_2|$. Applying the law of cosines to the angles at B_1 we get $\beta_1 \leq \beta$. Exchanging the roles of β_1 and β_2 in the above argumentation we have $\beta_2 \leq \gamma$. \square

Theorem 1 (Basic Gluing Theorem). ([1], Chapter II.11.1) Let X_1 and X_2 be CAT(0) spaces and let A be a complete metric space. If a convex subspace A_i of X_i , $i = 1, 2$, is isometric to A then $X_1 \amalg_A X_2$ is a CAT(0) space.

Proof. By Lemma 2.13, the gluing $X = X_1 \amalg_A X_2$ is a geodesic space. We shall verify the angle criterion for CAT(0) spaces given in Proposition 2.6. Since the other cases are obvious, it suffices to consider geodesic triangles $\Delta = \Delta(x_1, x_2, y)$ with vertices $x_1 \in X_1 \setminus A_1$ and $x_2 \in X_2 \setminus A_2$. Without loss of generality we can assume that $y \in X_1$.

Fix points $z, z' \in A_1$ with $z \in [x_1, x_2]$ and $z' \in [y, x_2]$. Note that $[x_1, z], [y, z'] \subset X_1$, $[z, x_2], [z', x_2] \subset X_2$ and $[z, z'] \subset A_1$. Moreover, $[x_1, z] \subset [x_1, x_2]$ and $[z', x_2] \subset [y, x_2]$, so the Alexandrov angles $\angle([x_2, x_1], [x_2, y])$ in X and $\angle([x_2, z], [x_2, z'])$ in X_2 are equal. Similarly, $\angle([y, x_1], [y, x_2])$ in X is equal to $\angle([y, x_1], [y, z'])$ in X_1 .

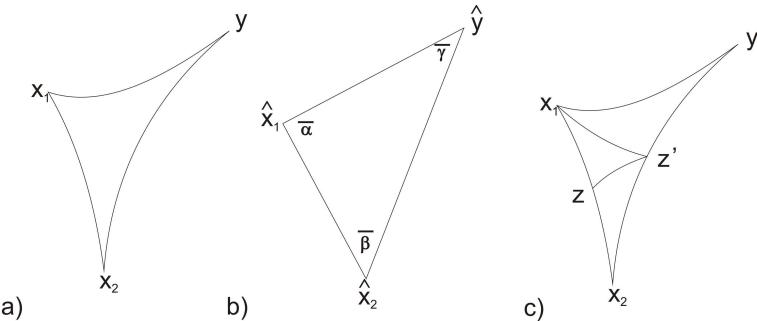


Fig. 5. a) Geodesic triangle Δ , b) its comparison triangle $\bar{\Delta}$, c) triangles $\Delta_1, \Delta_2, \Delta_3$

First consider the geodesic triangles $\Delta_i = \Delta(x_i, z, z') \subset X_i$, $i \in \{1, 2\}$, and denote by $\alpha_i, \beta_i, \gamma_i$ the angles od Δ_i at z, x_i, z' , respectively.

Since X_1, X_2 are CAT(0) spaces, the angles of comparison triangles $\bar{\Delta}_i = \Delta(\bar{x}_i, \bar{z}, \bar{z}')$, $i \in \{1, 2\}$, satisfy the following inequalities:

$$\alpha_i \leq \bar{\alpha}_i, \quad \beta_i \leq \bar{\beta}_i, \quad \gamma_i \leq \bar{\gamma}_i. \quad (20)$$

Take the geodesic triangle $\Delta_3 = \Delta(x_1, x_2, z')$ with angles $\delta_1, \delta_2, \varepsilon$ at the vertices x_1, x_2, z' . We claim that the angles $\delta_1, \delta_2, \varepsilon$ of Δ_3 are no greater than the corresponding angles $\bar{\delta}_1, \bar{\delta}_2, \bar{\varepsilon}$ of its comparison triangle $\bar{\Delta}_3 = \Delta(\tilde{x}_1, \tilde{x}_2, \tilde{z}')$ in \mathbb{R}^2 .

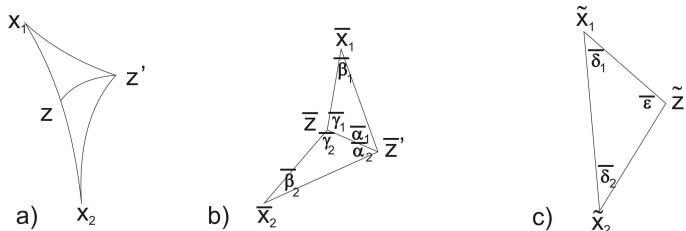


Fig. 6. a) Triangles Δ_1 and Δ_2 , b) comparison triangles $\bar{\Delta}_1$ and $\bar{\Delta}_2$, c) comparison triangle $\bar{\Delta}_3$

Observe that

$$\varepsilon \leq \alpha_1 + \alpha_2, \quad \beta_1 = \delta_1, \quad \beta_2 = \delta_2, \quad \gamma_1 + \gamma_2 \geq \pi. \quad (21)$$

By Alexandrov's lemma

$$\overline{\alpha}_1 + \overline{\alpha}_2 \leq \overline{\varepsilon}, \quad \overline{\beta}_1 \leq \overline{\delta}_1, \quad \overline{\beta}_2 \leq \overline{\delta}_2 \quad (22)$$

and hence, by (20) and (21),

$$\varepsilon \leq \overline{\varepsilon}, \quad \delta_1 \leq \overline{\delta}_1, \quad \delta_2 \leq \overline{\delta}_2. \quad (23)$$

Now consider the geodesic triangle $\Delta_4 = \Delta(x_1, y, z') \subset X_1$ and its comparison triangle $\overline{\Delta}_4 = \Delta(\tilde{x}_1, \tilde{y}, \tilde{z}')$. Let φ, ψ, η be the angles of Δ_4 at the vertices x_1, y, z' . Denote the corresponding angles in $\overline{\Delta}_4$ by $\overline{\varphi}, \overline{\psi}, \overline{\eta}$. Then

$$\varphi \leq \overline{\varphi}, \quad \psi \leq \overline{\psi}, \quad \eta \leq \overline{\eta}. \quad (24)$$

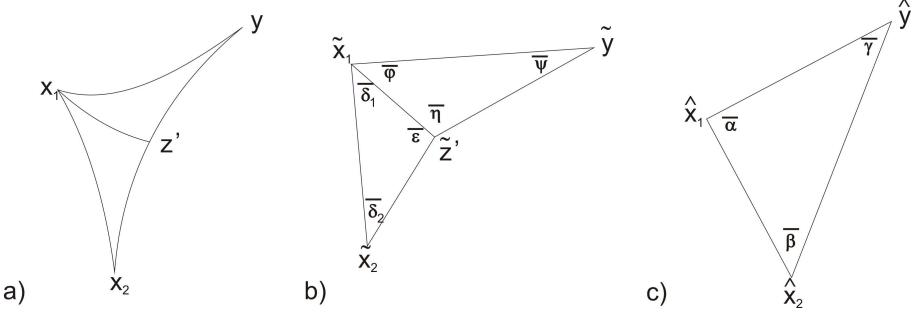


Fig. 7. a) Triangles Δ_3 and Δ_4 , b) comparison triangles $\overline{\Delta}_3$ and $\overline{\Delta}_4$, c) comparison triangle $\overline{\Delta}$

Finally, let $\overline{\Delta} = \Delta(\widehat{x_1}, \widehat{x_2}, \widehat{y})$ be the comparison triangle for $\Delta = \Delta(x_1, x_2, y)$. Denote by α, β, γ (resp. $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$) the angles of Δ (resp. $\overline{\Delta}$) at the vertices x_1, x_2, y (resp. $\widehat{x_1}, \widehat{x_2}, \widehat{y}$). Note that

$$\alpha \leq \delta_1 + \varphi, \quad \beta = \beta_2 = \delta_2, \quad \gamma = \psi, \quad \varepsilon + \eta \geq \pi. \quad (25)$$

Applying Alexandrov's lemma to $\overline{\Delta}_3$ and $\overline{\Delta}_4$ we get

$$\overline{\delta}_1 + \overline{\varphi} \leq \overline{\alpha}, \quad \overline{\delta}_2 \leq \overline{\beta}, \quad \overline{\psi} \leq \overline{\gamma}. \quad (26)$$

By (23), (24) and (25) we obtain $\alpha \leq \overline{\alpha}, \beta \leq \overline{\beta}, \gamma \leq \overline{\gamma}$. \square

Theorem 2. *Let $(X_\lambda, d_\lambda)_{\lambda \in \Lambda}$ be a family of CAT(0) spaces with convex subspaces $A_\lambda \subset X_\lambda$ isometric to a complete space A . Then the space $X = \coprod_A X_\lambda$ obtained by gluing X_λ along A is a CAT(0) space.*

Proof. First observe that for all $\lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2, x \in X_{\lambda_1}, y \in X_{\lambda_2}$, the geodesic segment $[x, y]$ in X is contained in $X_{\lambda_1} \sqcup_A X_{\lambda_2}$. Suppose that there exist $\lambda_3 \in \Lambda$,

$\lambda_1 \neq \lambda_3 \neq \lambda_2$, and $z \in X_{\lambda_3}$ such that $z \in [x, y]$. Then $d(x, y) = d(x, z) + d(z, y)$. According to Lemma 2.13, there exist $a_1, a_2, a_3 \in A$ such that

$$\begin{aligned} d(x, y) &= d(x, a_1) + d(a_1, y), \\ d(x, z) &= d(x, a_2) + d(a_2, z), \\ d(y, z) &= d(z, a_3) + d(a_3, y). \end{aligned}$$

If $z \notin A$ then $d(a_2, a_3) < d(a_2, z) + d(z, a_3)$ and hence

$$\begin{aligned} d(x, y) &= d(x, z) + d(z, y) = d(x, a_2) + d(a_2, z) + d(z, a_3) + d(a_3, y) \\ &> d(x, a_2) + d(a_2, a_3) + d(a_3, y) \geq d(x, a_3) + d(a_3, y) \geq d(x, y). \end{aligned}$$

Therefore any geodesic triangle Δ in X with vertices $x_i \in X_{\lambda_i}$, $i \in \{1, 2, 3\}$, is contained in $(X_{\lambda_1} \sqcup_A X_{\lambda_2}) \sqcup_A X_{\lambda_3}$. By Theorem 1 both $X_{\lambda_1} \sqcup_A X_{\lambda_2}$ and $(X_{\lambda_1} \sqcup_A X_{\lambda_2}) \sqcup_A X_{\lambda_3}$ are CAT(0) spaces, so the (Alexandrov) angles of Δ in X are no greater than the corresponding angles of its comparison triangle in \mathbb{R}^2 . This proves that X is a CAT(0) space. \square

Remark 2.21. In fact, Theorem 2 is a part of a more general theorem formulated and proven for all numbers $\kappa \in \mathbb{R}$. Namely, if each X_λ is a CAT(κ) space (with the same number κ for each λ) then the gluing X is also a CAT(κ) space. For the details see [1], Chapter II.11, Theorem 11.3.

3. Gluing along a point

Assume that all X_λ , $\lambda \in \Lambda$, are uniquely geodesic spaces. To obtain their gluing along a point take $A = \{\theta\}$, in each X_λ choose a point θ_λ corresponding to θ and put $A_\lambda = \{\theta_\lambda\}$. Then $X = \coprod_\theta X_\lambda$ is the gluing of X_λ along the singleton $\{\theta\}$. Clearly, X is also a uniquely geodesic space.

Theorem 3. Let X_λ , $\lambda \in \Lambda$, be Busemann spaces. Then the gluing

$$X = \coprod_\theta X_\lambda$$

of X_λ along θ is also a Busemann space.

Proof. Let $x, y, z \in X$. To show that X is a Busemann space it suffices to check that x, y and z satisfy the condition (3), i.e. that if x' is a midpoint of the metric segment $[x, z]$ and y' is a midpoint of $[y, z]$, then

$$d(x', y') \leq \frac{1}{2}d(x, y). \quad (27)$$

We consider three separate cases.

- I. Let us suppose that x, y, z belong to different sets $X_\lambda \setminus \{\theta\}$. Then the union $[\theta, x] \cup [\theta, y] \cup [\theta, z]$ is a metric tree and inequality (27) follows directly from its properties (see Example 2.7).

- II. Let us choose $x, z \in X_\lambda \setminus \{\theta\}$ and $y \in X_{\lambda'}, \lambda \neq \lambda'$. Then the Busemann property of X_λ implies that

$$d\left(x', \frac{1}{2}z + \frac{1}{2}\theta\right) \leq \frac{1}{2}d(x, \theta).$$

At the same time

$$d\left(\frac{1}{2}z + \frac{1}{2}\theta, y'\right) = \frac{1}{2}d(y, z) - \frac{1}{2}d(z, \theta) = \frac{1}{2}d(\theta, y),$$

from which it follows that

$$d(x', y') \leq \frac{1}{2}(d(x, \theta) + d(\theta, y)) = \frac{1}{2}d(x, y).$$

- III. Let us choose $x, y \in X_\lambda \setminus \{\theta\}$ and $z \in X_{\lambda'}, \lambda \neq \lambda'$.

- a) First let us assume that $x', y' \in [z, \theta]$. Hence

$$d(x', \theta) = d(z, \theta) - \frac{1}{2}d(z, x),$$

$$d(y', \theta) = d(z, \theta) - \frac{1}{2}d(z, y)$$

and

$$d(x', y') = |d(x', \theta) - d(y', \theta)| = \frac{1}{2}|d(z, x) - d(z, y)|,$$

which proves (27).

- b) Now let us suppose that $x' \in [z, \theta]$ and $y' \notin [z, \theta]$. Therefore

$$d(x', \theta) = d(z, \theta) - \frac{1}{2}d(z, x),$$

$$d(y', \theta) = \frac{1}{2}d(z, y) - d(z, \theta)$$

and

$$d(x', y') = d(x', \theta) + d(y', \theta) = \frac{1}{2}(d(z, y) - d(z, x)).$$

- c) Here we consider the case where both x' and y' do not belong to $[z, \theta]$. Hence

$$\frac{1}{2} = \frac{d(z, \theta) + d(\theta, x')}{d(z, \theta) + d(\theta, x)},$$

so

$$d(\theta, x') = \frac{1}{2}d(\theta, x) - \frac{1}{2}d(z, \theta) = sd(\theta, x),$$

where $s = \frac{1}{2} - \frac{d(\theta, z)}{2d(\theta, x)}$. Thus, $x' = (1-s)\theta + sx$ and the Busemann property implies that

$$d((1-s)\theta + sx, (1-s)\theta + sy) \leq sd(x, y) = \frac{1}{2}d(x, y) - \frac{d(\theta, z)}{2d(\theta, x)}d(x, y).$$

But at the same time

$$d(y', \theta) = \frac{1}{2}d(\theta, y) - \frac{1}{2}d(\theta, z)$$

and

$$d((1-s)\theta + sy, \theta) = sd(\theta, y) = \left(\frac{1}{2} - \frac{d(\theta, z)}{2d(\theta, x)} \right) d(\theta, y),$$

which yields

$$\begin{aligned} d(y', (1-s)\theta + sy) &= \left| \frac{1}{2}d(\theta, z) - \frac{d(\theta, z)}{2d(\theta, x)}d(\theta, y) \right| \\ &= \frac{1}{2}d(\theta, z) \frac{|d(\theta, x) - d(\theta, y)|}{d(\theta, x)} \leq \frac{1}{2}d(\theta, z) \frac{d(x, y)}{d(\theta, x)}. \end{aligned}$$

This completes the proof of (27) in this case.

The remaining cases can be treated as obvious. \square

4. Planting a tree on Busemann ground

Theorem 4.1. *Let M be a metric tree, X be a Busemann space and A be a metric segment, isometrically embedded as A_M in M and as A_X in X . Then $M \amalg_A X$ also is a Busemann space.*

Proof. First we will prove that $M \amalg_A X$ is uniquely geodesic. Assume that $x \in M \setminus A_M$ and $y \in X \setminus A_X$. Clearly, the existence of any geodesic joining x and y follows from compactness of a geodesic segment. So let

$$d(x, y) = d(x, a) + d(a, y) \quad \text{and} \quad d(x, y) = d(x, b) + d(b, y). \quad (28)$$

Since M is a metric tree there is a point $u \in [a, b]$ such that

$$d(a, u) + d(u, x) = d(a, x) \quad \text{and} \quad d(b, u) + d(u, x) = d(b, x).$$

If $u \neq a$ and $u \neq b$ then

$$d(u, y) \leq \max\{d(a, y), d(b, y)\}$$

and assuming that this maximum is equal to $d(a, y)$ we get

$$d(x, y) \leq d(x, u) + d(u, y) < d(x, a) + d(a, y) = d(x, y),$$

a contradiction.

Now let $u = a$. From (28) it follows that

$$d(x, y) = d(x, a) + d(a, b) + d(b, y),$$

which yields

$$d(a, y) = d(a, b) + d(b, y)$$

and since X is also uniquely geodesic as a Busemann space the proof is complete.

Note that, again by compactness of A , for every point x of $M \amalg_A X$ there exists a point $a \in A$ such that $d(a, x) = \inf_{y \in A} d(x, y)$.

It remains to show that $M \amalg_A X$ is a Busemann space. Let x, y, z be three distinct points in $M \amalg_A X$ and let y', z' be the midpoints of the geodesic arcs $[x, y]$, $[x, z]$ in $M \amalg_A X$. We have to verify that

$$d(y', z') \leq \frac{1}{2}d(y, z). \quad (29)$$

All the cases where only one of the points x, y, z belongs to the metric tree M are contained in Theorem 3, so let us focus on the cases with two points in M . We will consider them separately.

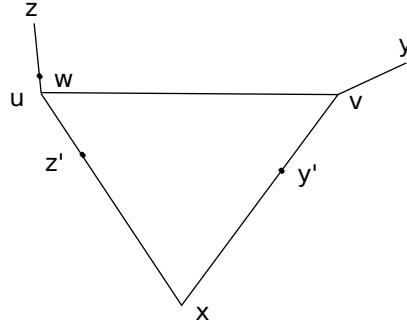


Fig. 8. Geodesic triangle $\Delta(x, y, z)$ with $x \in X \setminus A_X$ and $y, z \in M \setminus A_M$

First let us assume that $y, z \in M$ but $y', z' \in X$. Then one can find u and v in A such that for each $t \in X$ we have

$$d(y, t) = d(y, v) + d(v, t) \quad \text{and} \quad d(z, t) = d(z, u) + d(u, t).$$

Let us assume additionally that

$$\frac{d(y, v)}{d(y, x)} \leq \frac{d(z, u)}{d(z, x)}.$$

So there is $w \in [u, z]$ such that

$$\frac{d(z, w)}{d(z, x)} = \frac{d(y, v)}{d(y, x)}.$$

Theorem 3 applied to $X \amalg_{\{u\}} [x, u]$ yields

$$d(y', z') \leq d(w, v) \cdot \frac{d(x, z')}{d(x, w)}.$$

From the basic properties of M we get

$$d(w, v) = d(y, z) - d(w, z) - d(v, y),$$

which leads to

$$d(y', z') \leq \frac{d(x, z')}{d(x, w)} (d(y, z) - d(w, z) - d(v, y)).$$

At the same time

$$\begin{aligned} -d(w, z) \cdot \frac{d(x, z')}{d(x, w)} - d(v, y) \cdot \frac{d(x, z')}{d(x, w)} &= -d(w, z) \cdot \frac{d(x, z')}{d(x, w)} - d(v, y) \cdot \frac{d(x, y')}{d(x, v)} \\ &= -d(x, z') \cdot \frac{d(w, z)}{d(x, w)} - d(x, y') \cdot \frac{d(v, y)}{d(x, v)} = -\frac{d(w, z)}{d(x, w)} (d(x, y') + d(x, z')). \end{aligned}$$

Thus

$$d(y', z') \leq d(y, z) \cdot \frac{d(x, z')}{d(x, w)} - \frac{d(w, z)}{d(x, w)} \cdot d(y', z'),$$

which yields

$$d(y', z') \left(\frac{d(x, w) + d(w, z)}{d(x, w)} \right) \leq d(y, z) \cdot \frac{d(x, z')}{d(x, w)}$$

and completes the proof for this case.

If one of points y' or z' belongs to $M \setminus A_M$ then the proof goes with the same pattern. The case $y', z' \in M \setminus A_M$ is obvious.

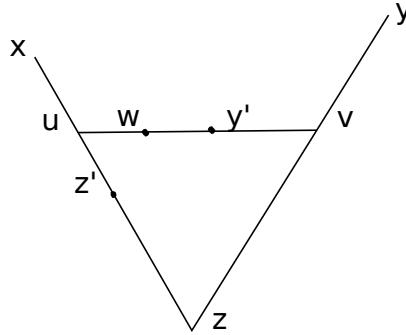


Fig. 9. Geodesic triangle $\Delta(x, y, z)$ with $z \in X \setminus A_X$ and $x, y \in M \setminus A_M$

Next let us suppose that $x, y \in M \setminus A_M$ and $z \in X \setminus A_X$. Obviously there are two points $u, v \in A_M$ such that

$$d(z, u) + d(u, x) = d(z, x) \quad \text{and} \quad d(y, v) + d(v, z) = d(y, z). \quad (30)$$

If y' or z' belongs to $[x, u]$ then following the proof of Theorem 3 we obtain (29). Therefore let us assume that both points are in X . Then one may find $w \in [x, v]$ in such a way that

$$d(x, w) = \frac{1}{2}d(x, v).$$

Again proceeding as in the proof of Theorem 3 we get

$$d(z', w) \leq \frac{1}{2}d(z, v).$$

At the same time let us notice that

$$d(w, y') = \frac{1}{2}d(v, y)$$

and the final inequality follows from (30).

If $z' \in [u, z]$ and $y' \in [v, y]$ then applying the arguments similar to the ones used in part III c) of the proof of Theorem 3 we obtain

$$d(z', v) \leq \frac{1}{2}(d(z, v) + d(u, v) + d(x, u)),$$

which implies (29).

Clearly, if $u = v$, then there must be a unique point $w \in [x, y]$ such that

$$d(x, u) = d(x, w) + d(w, u) \quad \text{and} \quad d(y, u) = d(y, w) + d(w, u)$$

and the triangle $\Delta(x, y, z)$ consists of three common-ended geodesic segments $[x, w]$, $[y, w]$ and $[z, w]$ which form a metric tree. In such a case the final result follows from the fact that each metric tree is a Busemann space. \square

5. Counterexamples

Example 5.1. Let X be the plane \mathbb{R}^2 with the Euclidean metric and let Y be the same space \mathbb{R}^2 equipped with the norm from l_3 , i.e., $\|y\|_3 = \sqrt[3]{|y_1|^3 + |y_2|^3}$. We glue this spaces with respect to the closed convex subset

$$A = A_X = A_Y = \{(x, 0) | x \in \mathbb{R}\},$$

where A_X and A_Y are considered as subsets of X and Y , respectively.

Let $x = (1, -1)$, $y = (-1, -1)$ belong to X and let $z = (0, 1)$ be a point in Y . Then

$$\begin{aligned} d(x, z) &= d(y, z) = \min_{s \in (0, 1)} d(x, (s, 0)) + d((s, 0), z) \\ &= \min_{s \in (0, 1)} \sqrt{(1-s)^2 + 1} + \sqrt[3]{s^3 + 1}. \end{aligned}$$

Easy computations show that this function attains its minimum at s if and only if

$$\frac{1-s}{\sqrt{1+(1-s)^2}} = \frac{s^2}{(1+s^3)^{2/3}}.$$

The latter condition leads to the 16th degree polynomial equation which cannot be solved with analytical methods. Its numerical solution was obtained by the Mathe-

matica software and

$$s \approx 0.632938192307414.$$

Putting $u = (s, 0)$, $v = (-s, 0)$ we have $u, v \in A_X = A_Y$ and

$$d(x, z) = d(x, u) + d(u, z), \quad d(y, z) = d(y, v) + d(v, z), \quad d(x, u) = d(y, v).$$

Clearly, we cannot deduce that u is the midpoint of $[x, z]$, therefore we check whether the Busemann condition (1) holds for $x_0 = y_0 = z$, $x_1 = x$, $y_1 = y$, $t = \frac{d(z, u)}{d(x, z)}$ and $x_t = u$, $y_t = v$. We may estimate

$$d(x, u) \approx \sqrt{1 - 0.632938192307414^2 + 1} \approx 1.065239114314973$$

and

$$d(u, z) \approx \sqrt[3]{0.632938192307414^3 + 1} \approx 1.0782395436939778.$$

Hence

$$t = \frac{d(z, u)}{d(x, z)} = \frac{d(z, v)}{d(y, z)} \approx 0.503033.$$

At the same time

$$\frac{d(u, v)}{d(x, y)} = \frac{2s}{2} > 0.6 > t,$$

so $X \amalg_A Y$ is not a Busemann space.

Example 5.2. Let X , Y and A be the same as in Example 5.1 but this time we slightly change our approach to the problem. Let us choose $x = (2, -1)$ and $y = (-2, -1)$ as points of X . Our next step is to find $z \in Y$ in such a way that the midpoints x' and y' of the geodesic segments $[x, z]$ and $[y, z]$ belong (both) to A_X . Clearly, z must be of the form $(0, a)$, $x' = (t, 0)$, $y' = (-t, 0)$ for some $t \in (0, 2)$ and $d(z, x') = d(z, y') = 1/2 \cdot d(z, x)$. Then

$$d(z, x') = \sqrt[3]{t^3 + a^3}$$

and

$$d(x, x') = \sqrt{(2-t)^2 + 1}.$$

So we define the function

$$f(a) = \inf_{s \in (0, 2)} \left(\sqrt[3]{s^3 + a^3} + \sqrt{(2-s)^2 + 1} \right).$$

and we are looking for $a > 0$ and $t \in [0, 2]$ such that

$$\begin{cases} f(a) = \sqrt[3]{t^3 + a^3} + \sqrt{(2-t)^2 + 1} \\ \sqrt[3]{t^3 + a^3} = \sqrt{(2-t)^2 + 1} \end{cases}$$

Using again the Mathematica Software we get the estimated solutions $a \approx 1.033113$ and $t \approx 1.1002002$. This leads to the inequality

$$\frac{d(x', y')}{d(x, y)} = \frac{2t}{4} > 0.55,$$

which again contradicts the fact that $X \amalg_A Y$ is a Busemann space.

Example 5.3. Changing the roles of X and Y in the previous example we obtain that

$$\frac{d(x', y')}{d(x, y)} = \frac{2t}{4} < 0.5.$$

Remark 5.4. Note that the spaces X and Y considered in the above examples are linear and their closed convex subspaces $A_X = A_Y$ are proper. Therefore $X \amalg_A Y$ is uniquely geodesic (see Corollary 2.19) but fails to be a Busemann space.

Example 5.5. Replaceing \mathbb{R}^2 with $[-3, 3] \times [-3, 3]$ in Examples 5.1-5.2 and taking $A = A_X = A_Y = [-3, 3] \times \{0\}$ we get the amalgamation of two compact spaces along their closed convex subset which is a metric segment. Choosing the same points as before we get the identical results contradicting the Busemann condition (1).

We are convinced that the amalgamation $X \amalg_A Y$ fails to be a Busemann space whenever X is \mathbb{R}^2 with the norm $\|\cdot\|_p$, Y is \mathbb{R}^2 with the norm $\|\cdot\|_q$, $A = \mathbb{R} \times \{0\}$, $p, q \in (1, \infty)$, $p \neq q$. We suppose that for $p < q$ it is possible to find points $x = (a, 1)$, $y = (-a, 1) \in X$ and $z = (0, b) \in Y$ such that

$$d(x', y') > \frac{d(z, x')}{d(z, x)} d(x, y),$$

where $x' \in A \cap [x, z]$, $y' \in A \cap [y, z]$. The calculations are nevertheless rather complicated even in the relatively simple case of $p = 2$ and $q = 3$.

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